◎ Vector Spaces and Subspaces

$\mathbb{R}^n$ = all column vectors with n (real) components
$= \{(v_1, v_2, \cdots, v_n) : v_i \in \mathbb{R},\ i = 1, 2, \cdots, n\}$

$\begin{bmatrix} 4 \\ \pi \end{bmatrix} \in \mathbb{R}^2 \qquad (1, 1, 0, 1, 1) \in \mathbb{R}^5$

vector-addition $\quad \underline{v} + \underline{w} \qquad \begin{bmatrix} 2 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

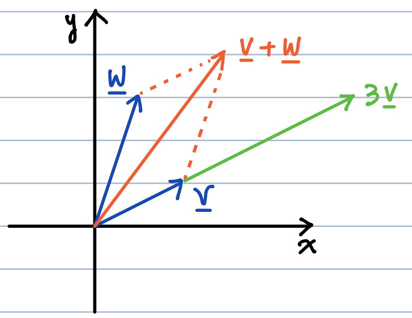
scalar multiplication $\quad c\,\underline{v} \qquad 2\begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 6 \\ 8 \end{bmatrix}$

scalar (real number)

If $\underline{v}, \underline{w} \in \mathbb{R}^n$, then $\underline{v} + \underline{w} \in \mathbb{R}^n$
If $\underline{v} \in \mathbb{R}^n$, $c \in \mathbb{R}$, then $c\underline{v} \in \mathbb{R}^n$

$\underline{0} = 0 \cdot \underline{v} \in \mathbb{R}^n$

$\mathbb{R}^2$



◎ vector space: (V: a set of vectors)

◎ two operations:

vector addition: $\underline{v} \in V,\ \underline{w} \in V \Rightarrow \underline{v} + \underline{w} \in V$
scalar multiplication: $\underline{v} \in V \Rightarrow c\underline{v} \in V$

◎ eight rules:

(1) $\underline{v} + \underline{w} = \underline{w} + \underline{v}$
(2) $\underline{u} + (\underline{v} + \underline{w}) = (\underline{u} + \underline{v}) + \underline{w}$
(3) There is a unique "zero vector" $\underline{0}$ such that $\underline{v} + \underline{0} = \underline{v}$ for all $\underline{v} \in V$
(4) For each $\underline{v}$, there is a unique vector $-\underline{v}$ such that $\underline{v} + (-\underline{v}) = \underline{0}$
(5) $1 \cdot \underline{v} = \underline{v}$
(6) $(c_1 c_2)\underline{v} = c_1(c_2 \underline{v})$
(7) $c(\underline{v} + \underline{w}) = c\underline{v} + c\underline{w}$
(8) $(c_1 + c_2)\underline{v} = c_1\underline{v} + c_2\underline{v}$

☆ **Example**

1. $\mathbb{R}^n$ is a vector space

2. $M = \{$ all real 2x2 matrices $\}$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 3 & 5 \end{bmatrix} \in M$$

$$4 \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 4 & 8 \\ 12 & 16 \end{bmatrix} \in M$$

$\Rightarrow$ M is a vector space.

3. $F = \{$ all real functions $f(x) \}$

$f(x) + g(x) \in F$

$c f(x) \in F$

$\Rightarrow$ F is a vector space.

4. $Z = \{ \underline{0} \}$

$\underline{0} + \underline{0} = \underline{0} \in Z$

$c \cdot \underline{0} = \underline{0} \in Z$

$\Rightarrow$ Z is a vector space.

Remark $(0+1)\underline{v} = 1 \cdot \underline{v} = \underline{v}$

$\qquad \|$ $\qquad\qquad \Rightarrow \quad 0 \cdot \underline{v} = \underline{0}$

$0 \cdot \underline{v} + 1 \cdot \underline{v} = 0 \cdot \underline{v} + \underline{v}$

Remark $\underline{0} = 0 \cdot \underline{v} = (1 + (-1))\underline{v} = \underline{v} + (-1)\underline{v} \Rightarrow (-1)\underline{v} = -\underline{v}$

Def A subset W of a vector space V is a subspace if W itself is a vector space.

Def A subset W of a vector space V is a subspace if

(i) $\underline{v} \in W, \ \underline{w} \in W \Rightarrow \underline{v} + \underline{w} \in W$

(ii) $\underline{v} \in W \Rightarrow c\underline{v} \in W$ for any c.

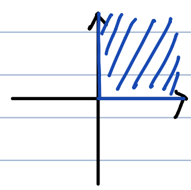
Claim Every subspace contains the zero vector.

Proof $0 \cdot \underline{v} = \underline{0} \in W$. ∎

☆ **Example**

1. $U = \{ (x, y) : x \geq 0, \ y \geq 0 \}$

Is U a subspace?



No since $-1(1, 0) = (-1, 0) \notin U$

even if $(1, 0) \in U$

2. $M = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a, b, c, d \in \mathbb{R} \right\}$

$U = \left\{ \begin{bmatrix} a & b \\ 0 & d \end{bmatrix} : a, b, d \in \mathbb{R} \right\}$

If $A, B \in U$, $A + B \in U$

If $A \in U$, $cA \in U$

$\therefore U$ is a subspace of M.

◎ Column Space

Def The column space $C(A)$ of a matrix A consists of all linear combinations of the columns of A.

☆ Example

$$A = \begin{bmatrix} 1 & 0 \\ 4 & 3 \\ 2 & 3 \end{bmatrix}$$

$$C(A) = \left\{ x_1 \begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 3 \\ 3 \end{bmatrix} : x_1, x_2 \in \mathbb{R} \right\}$$

$$= \left\{ A\underline{x} : \underline{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^2 \right\}$$

$A\underline{x} = \underline{b}$

$$\Leftrightarrow x_1 \begin{bmatrix} \underline{a_1} \end{bmatrix} + x_2 \begin{bmatrix} \underline{a_2} \end{bmatrix} + \cdots + x_n \begin{bmatrix} \underline{a_n} \end{bmatrix} = \underline{b}$$

$\therefore$ The system $A\underline{x} = \underline{b}$ is solvable iff $\underline{b} \in C(A)$.

☆ Example

What are the column spaces of

① $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, ② $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$, ③ $B = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 4 \end{bmatrix}$?

Ans:

① $x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^2$

$\therefore C(I) = \mathbb{R}^2$

② $x_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 4 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + 2x_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} = (x_1 + 2x_2) \begin{bmatrix} 1 \\ 2 \end{bmatrix} = x \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$\therefore C(A) = \left\{ x \begin{bmatrix} 1 \\ 2 \end{bmatrix} : x \in \mathbb{R} \right\}$

③ $x_1\begin{bmatrix}1\\0\end{bmatrix}+x_2\begin{bmatrix}2\\0\end{bmatrix}+x_3\begin{bmatrix}3\\4\end{bmatrix}$

$=(x_1+2x_2)\begin{bmatrix}1\\0\end{bmatrix}+x_3\begin{bmatrix}3\\4\end{bmatrix}$

$=x_4\begin{bmatrix}1\\0\end{bmatrix}+x_3\begin{bmatrix}3\\4\end{bmatrix}$

$\begin{bmatrix}1&3\\0&4\end{bmatrix}\begin{bmatrix}x_4\\x_3\end{bmatrix}=\begin{bmatrix}b_1\\b_2\end{bmatrix}$ is always solvable for any b_1, b_2

$\therefore\ \mathcal{C}(B)=\mathbb{R}^2.$

$\Rightarrow$ All of them are subspaces of $\mathbb{R}^2$.

<u>Claim</u> If A is $m\times n$, then $\mathcal{C}(A)$ is a subspace of $\mathbb{R}^m$.

S = set of vectors in a vector space V (probably not a subspace)
SS = the set of all linear combinations of vectors in S

We call SS the "span" of S.

Then SS is a subspace of V, called the subspace "spanned" by S.

☆ <u>Example</u>

S = the set of columns of A
$A=\begin{bmatrix}1&2\\2&3\end{bmatrix}$
SS = the column space of A = $\mathcal{C}(A)$